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IMPROVED FORM OF SIMPSON'S ONE-THIRD RULE FOR FINDING APPROXIMATE VALUE OF DEFINITE INTEGRALS BY USING TRIGONOMETRIC FUNCTIONS

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Abstract
Numerical integration plays an important role in various fields of science and engineering. Existing numerical integration methods are midpoint rule, Trapezoidal rule, Simpson's rule, Rom berg's and Boole's rule. In this paper I have derived a new form of numerical integration by using the condition on the integrand $f(x)$ satisfies the condition $f(x_i) = \emptyset(x_i), i = 0, 1, 2 \ldots n$ where $\emptyset(x) = \mathrm{A}\cos x + B\, sin\, x + Cx$. For this method I choose n value so that step size h is sufficiently small. As far as novelty of the paper is concern, fewer calculations taken into account with enough preciseness and results are in agreement with Simpson's one-third rule.
Keywords: Numerical integration; Trigonometric functions; Quadratic polynomial; Simpson's rule; Trapezoid's rule; approximations.

抽象的

数值积分在科学和工程的各个领域都发挥着重要作用。 现有的数值积分方法有中点法则、梯形法则、辛普森法则、罗姆伯格法则和布尔法则。 在本文中，我通过使用被积函数 f(x) 上的条件推导出一种新形式的数值积分 满足条件 f(x_i)=∅(x_i),i=0,1,2…n 其中 ∅(x) =〖A cos〗 x+B sin x+Cx。 对于这种方法，我选择 n 值，以便步长 h 足够小。 就论文的新颖性而言，考虑到足够精确的计算较少，结果符合辛普森的三分之一规则。

关键词：数值积分； 三角函数； 二次多项式； 辛普森法则； 梯形法则； 近似值。

1. Introduction :

In Numerical integration we have several methods like Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule, Milne's rule, and Boole's rule [1–7] and [8].

Simpson's One-Third Rule:

Using Midpoint rule, we approximate the area under the curve by using rectangles. We approximated the curve with piece wise constant functions. In the Trapezoidal rule, we find the approximate value of definite integral using piece wise linear functions. In the Simpson's rule we are using piece wise quadratic functions. We divide the interval $[a, b]$ into an even number of sub intervals each of equal width.

$I = \int_a^b f(x)dx$,

$h = \frac{b-a}{n}$ and $x_0 = a$, $x_1 = a + h$, $x_{2,} = a + 2h$, …. $x_n = b$

Over the first pair of sub intervals we approximate

$$\int_{x_0}^{x_2} f(x)dx \text{ with } \int_{x_0}^{x_2} P_2(x)dx$$

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Where $P_2(x) = Ax^2 + Bx + C$ is the quadratic function passing through the points $\left(x_{0,}f(x_{0,})\right), \left(x_{1,}f(x_1)\right)$ and $\left(x_{2,}f(x_2)\right)$.

Over the next pair of sub intervals we approximate $\int_{x_2}^{x_4} f(x)dx$

$\left(x_{2,}f(x_{2,})\right), \left(x_{3,}f(x_3)\right)$ and $\left(x_{4,}f(x_4)\right)$

This process is continued with each successive pair of sub intervals. With Simpson's rule, we can approximate a definite integral by integrating a piece wise quadratic function.

To understand this formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two sub intervals. As we go through the derivation, we need to keep in mind the following relations.

$f(x_0) = P_2(x_0) = Ax_0{}^2 + Bx_0 + C$

$f(x_1) = P_2(x_1) = Ax_1{}^2 + Bx_1 + C$

$f(x_2) = P_2(x_2) = Ax_2{}^2 + Bx_2 + C$

$x_1 - x_0 = x_2 - x_1 = h$ and $x_1 = \frac{x_0+x_2}{2}$

Thus $\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} P_2(x)dx$

$\int_{x_0}^{x_2} P_2(x)dx = \int_{x_0}^{x_2}(Ax^2 + Bx + C\ )dx$

$= \left[A\frac{x^3}{3} + B\frac{x^2}{2} + Cx\right]_{x_0}^{x_2}$

$= \left[\frac{A}{3}(x_2{}^3 - x_0{}^3) + \frac{B}{2}(x_2{}^2 - x_0{}^2) + C(x_2 - x_0)\right]$

$= \left[\frac{A}{3}(x_2 - x_0)(x_2{}^2 + x_2x_0 + x_0{}^2) + \frac{B}{3}(x_2 - x_0)(x_2 + x_0) + C(x_2 - x_0)\right]$

Factor out $\frac{x_2-x_0}{6}$

$= \frac{(x_2-x_0)}{6}[2A(x_2{}^2 + x_2x_0 + x_0{}^2) + 3B(x_2 + x_0) + 6C(x_2 - x_0)]$

Re write the terms

$$= \frac{h}{3}\left[\left(Ax_2{}^2 + Bx_2 + C\right) + \left(Ax_2{}^2 + Bx_2 + C\right) + A(x_2{}^2 + 2x_2x_0 + x_0{}^2) + 2B(x_2 + x_0) + 4C\right]$$

$$= \frac{h}{3}[f(x_2) + f(x_0) + A(x_2 + x_0)^2 + 2B(x_2 + x_0) + 4C]$$

Replace $x_1 = \frac{x_0+x_2}{2}$

$$= \frac{h}{3}\left[f(x_2) + f(x_0) + A4x_1{}^2 + 4Bx_1 + 4C\right]$$

$$= \frac{h}{3}[f(x_2) + f(x_0) + 4f(x_1)]$$

$$\Rightarrow \int_{x_0}^{x_2} P_2(x)dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$\Rightarrow \int_{x_0}^{x_2} f(x)dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

Similarly we approximate

$\int_{x_2}^{x_4} f(x)dx \approx \frac{h}{3}[f(x_2) + 4f(x_3) + f(x_4)]$

And so on .....

$\int_{n-2}^{x_n} f(x)dx \approx \frac{h}{3}[f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$

Combining all these approximations, we get

$\int_a^b f(x)dx \approx S_n = \frac{h}{3}\left[f(x_0) + 4\left(f(x_1) + \cdots + f(x_{n-1})\right) + 2(\,f(x_2) + f(x_4) + \cdots f(x_{n-2})) + f(x_n)\right]$

2. Improved form of Simpson's rule:

To find the integral $\int_a^b f(x)dx$

Within some finite limits *a* and *b*. We replace $P_2(x)$ by a Linear combination $\Phi(x) = A\cos x + Bsin\ x + Cx$ passing through $\left(x_{0,}f(x_{0,})\right), \left(x_{1\,,}f(x_1)\right)$ and $\left(x_{2,}f(x_2)\right)$

Over the next pair of sub intervals we approximate $\int_{x_2}^{x_4} f(x)dx$

$\left(x_{2,}f(x_{2,})\right), \left(x_{3,}f(x_3)\right)$ and $\left(x_{4,}f(x_4)\right)$

This process continued with each successive pair of sub intervals. With improved form of Simpson's rule, I approximate a definite integral by integrating a piece wise linear function.

To understand the formula that I obtain for Simpson's rule, I begin by deriving a formula for this approximation over the first two sub intervals. As I go through the derivation, I need to keep in mind the following relations.

$$f(x_0) = \Phi(x_0) = Acos\ x_0 + B\ sin\ x_0 + C\ x_0$$

$$f(x_1) = \Phi(x_1) = Acos\ x_1 + B\ sinx_1 + Cx_1$$

$$f(x_2) = \Phi(x_2) = Acos\ x_2 + B \sin x_2 + Cx_2$$

Thus $\int_{x_0}^{x_2} f(x)dx \approx \int_{x_0}^{x_2} \Phi(x)dx$

$$\Rightarrow \int_{x_0}^{x_2} \Phi(x)dx = \int_{x_0}^{x_2}(A\cos x + Bsin\ x + Cx\ )dx$$

$$= \left[A \sin x - Bcos\ x + C\ \frac{x^2}{2}\right]_{x_0}^{x_2}$$

$$= \left[\ \left(A \sin x_2 - Bcos\ x_2 + C\ \frac{{x_2}^2}{2}\right) - \left(A \sin x_0 - Bcos\ x_0 + C\ \frac{{x_0}^2}{2}\right)\right]$$

$$\sin x_2$$

$$= A\ \ (\sin x_2 - \sin x_0) - B(\cos x_2 - \cos x_0) + C\ \frac{{x_2}^2 - {x_0}^2}{2}$$

$$= 2A \cos\left(\frac{x_2+x_0}{2}\right) \sin\left(\frac{x_2-x_0}{2}\right) + 2B \sin\left(\frac{x_2+x_0}{2}\right) \sin\left(\frac{x_2-x_0}{2}\right) + C\left(\frac{(x_2+x_0)(x_2-x_0)}{2}\right)$$

We know that $\frac{x_2+x_0}{2} = x_1$ and $\frac{x_2-x_0}{2} = h$

$$= 2A \cos(x_1) \sin(h) + 2B \sin(x_1) \sin(h) + C(2x_1h)$$

As $h \to 0$ $\quad \sin h \to h$

$$\int_{x_0}^{x_2} \Phi(x)dx = 2h(A \cos(x_1) + B \sin(x_1) + Cx_1)$$

$$\Rightarrow \int_{x_0}^{x_2} \Phi(x)dx = 2h\ f(x_1)$$

$$\Rightarrow \int_{x_0}^{x_2} f(x)dx \approx 2h\ f(x_1)$$

Similarly, I approximate

$$\int_{x_2}^{x_4} f(x)dx \approx 2h\ f(x_3)$$

And so on ……

$$\int_{x_{n-2}}^{x_n} f(x)dx \approx 2h\ f(x_{n-1})$$

Add all these approximations, we get

$$\int_a^b f(x)dx \approx S_n = 2h\ [f(x_1) + f(x_3) + \cdots + f(x_{n-1})]$$

Example 1: Evaluate $\int_0^2 \sin(\sqrt{x})\, dx$. Bu using $n = 40$

Solution: Here $f(x) = \sin(\sqrt{x})$ and $h = \frac{2-0}{40} = 0.05$

From the improved Simpson's rule

$$\int_a^b f(x)dx \approx 2h\ [f(x_1) + f(x_3) + \cdots + f(x_{n-1})]$$

$$\int_0^2 \sin(\sqrt{x})\, dx \approx 2h\ [f(x_1) + f(x_3) + \cdots + f(x_{39})]$$

$$\int_0^2 \sin(\sqrt{x})\, dx \approx 1.536350316$$

I compare this integral value by using different numerical integration methods from the following table.

Example 2: Evaluate $\int_0^1 e^{-\frac{x^2}{2}}\,dx$. Bu using $n = 40$

Solution: Here $f(x) = e^{-\frac{x^2}{2}}$ and $h = \frac{1-0}{40} = 0.025$

From the improved Simpson's rule

$$\int_a^b f(x)dx \approx 2h\,[f(x_1) + f(x_3) + \cdots + f(x_{n-1})]$$

$$\int_0^1 e^{-\frac{x^2}{2}}dx \approx 2h\,[f(x_1) + f(x_3) + \cdots + f(x_{39})]$$

$$\int_0^1 e^{-\frac{x^2}{2}}dx \approx 0.855687581$$

Results and Discussion:

We can observe the results from the above two examples and calculate the same numerical integration problems by using existing methods Trapezoidal's rule , Simpson's one-third rule, Simpson's three eighth rule, Milnes rule and Boole's rule

Following table-1 represents the different solutions of the numerical integration of $\int_0^2 \sin(\sqrt{x})\,dx$

S. No.	Method	Approximate value
1	Improved Simpson's 1/3 rule	1.536350316
2	Trapezoidal rule	1.532146139
3	Simpson's 1/3 rule	1.534135373
4	Simpson's 3/8 rule	1.534243708
5	Milnes rule	1.53455658
6	Boole's rule	1.53456859

Following table-2 represents the different solutions of the numerical integration of $I = \int_0^1 e^{-\frac{x^2}{2}}dx$

S. No.	Method	Approximate value
1	Improved Simpson's 1/3 rule	0.855687581
2	Trapezoidal rule	0.855634922
3	Simpson's 1/3 rule	0.855624392
4	Simpson's 3/8 rule	0.855624392
5	Milnes rule	0.855624392
6	Boole's rule	0.855624392

Conclusion: I analyze the study the numerical integration of Simpson's one third rule and try to modify the quadratic polynomial by using linear trigonometric function then observed the solutions ae simpler than accurate same as other methods. Error analysis need to study.

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