第 50 卷第 09 期
2023 年 9 月

湖南大学学报（自然科学版）
Journal of Hunan University（Natural Sciences）

Vol. 50. No. 09.
September 2023

Open Access Article

BOUNDS IN STRONG ROMAN DOMINATION

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Abstract
This article presents sharp lower and upper bounds for $\gamma_R(G)$ in term of diam (G). Recall that the eccentricity of vertex v in ecc (v) = max$\{d(u,w): w \in V\}$ and the diameter of G is diam (G) = max$\{$ ecc (v): $v \in V\}$. It has been assumed throughout this article that G is a nontrivial graph of order n≥ 2. 'Bounds on Roman domination number of a graph G containing cycles, in terms of its girth' has been presented. Recall that the girth of G (denoted by g(G)) is the length of the smallest cycle in G. Assume throughout this article that G is a non-trivial graph of order n ≥ 3 and contains a cycle.
Key Words: Roman domination, Strong Roman domination, Bounds.

Theorem 1
If a graph G has diameter three, then $\gamma_{SR}(G) \leq 3\delta$ Furthermore, this bound is sharp for infinite family of graphs.
Proof.
Since G has diameter three,
$N(u)$ dominates $V(G)$ for all vertex $u \in V(G)$.
Now, let $u \in V(G)$ and $deg(u) = \delta$.
Define $f: V(G) \to \{0,1,2,3\}$ by $f(x) = 3$ for$x \in N(u)$ and
$f(x) = 0$otherwise.
Obviously $f$is a strong roman domination function of G.
Thus $\gamma_{SR}(G) \leq 3\delta$.
To prove sharpness, let G be obtained from Cartesian product
$P_2 \square K_m \geq 4$ by adding a new vertex x and jointing it to exactly one vertex at each copy of K_m.
Obviously, $diam(G) = 3$ and $\gamma_{SR}(G) = 6 = 3\delta$.
This completes the proof.

Theorem 2
For a connected graph G, $\gamma_{SR}(G) \geq \lceil \frac{diam(G)+3}{2} \rceil$.
Furthermore, this bound is sharp for P_3 and P_4.
Proof.

Received: August 04, 2023 / Revised: August 30, 2023 / Accepted: September 18, 2023 / Published: September 30, 2023
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The statement is obviously true for K_3 .
Let G be a connected graph of order $n \geq 4$ and $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{SR}(G)$ −function.
Suppose that P = $v_1 v_2 . . . v_{diam(G)+1}$ is a diametral path in G.
This diametral path includes at most three edges from the induced subgraph $G[N[v]]$ for each $v \in V_1 \cup V_2 \cup V_3$.
Let $E' = \{v_i v_{i+1} \mid 1 \leq i \leq diam(G)\} \cap \cup_{v \in V1 \cup V2 \cup V3} E(G[N[v]])$.
Then the diametral path contains at most $|V_3| - 1$ edges not in E', joining the neighborhoods of the vertices of V_3.
Since G is a connected graph of order at least 4, $V_3 \neq \emptyset$.
Hence, $diam(G) \leq 2|V_2| + 2|V_1| + 2|V_3| + (|V_2| - 1)$

$\leq 2\gamma_{SR}(G) - 3$

$\gamma_{SR}(G) \geq \lceil \frac{diam(G)+3}{2} \rceil$

This completes the proof.

Theorem 3

For any connected graph G on n vertices $\gamma_{SR}(G) \leq n$.
Furthermore, this bound is sharp.

Proof.

Let $P = v_1 v_2 . . . v_{diam(G)+1}$ be a diametral path in G.
Moreover, let $f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{SR}(P)$- function.
By theorem K, the weight of f is $diam(G) + 1$.
Define $g: V(G) \to \{0,1,2,3\}$ by $g(x) = f(x)$ for $x \in V(P)$ and
$g(x) = 1$ for $x \in V(G)(P)$.
Obviously g is a strong roman domination function for G.
Hence, $\gamma_{SR}(G) \leq w(f) + (n - diam(G) - 1)$

$diam(G) + 1 + n - diam(G) - 1$

n

$\gamma_{SR}(G) \leq n$.

To prove sharpness,

Let G be obtained from a path $P = v_1 v_2 . . . v_{3k}$ $(k \geq 2)$
By adding a pendant edge v_{3u}.
Obviously, G achieves the bound

This completes the proof.

Theorem 4

For any connected graph G of order n with $\delta \geq 3$

$\gamma_{SR}(G) \leq n - (\delta - 2)$

Proof.

Let P= $v_1 v_2 . . . v_{diam(G)+1}$ be a diametral path in G and
$f = (V_0, V_1, V_2, V_3)$ be a $\gamma_{SR}(P) -$ function for which $|V_1|$ is minimized and V_2 is a $2 -$ packing.

Obviously, $|V_2|\ diam(G) + 1$
Let $V_2 = \{ u_1, . . . , u_k\}$ where $k = diam(G) + 1$
Since P is a diametral path, each vertex of V_2 has at least $\delta - 2$ neighbors in $V(G)(P)$ and $N(u_i) \cap N(u_j) = \emptyset$ if $u_i \neq u_j$.
Define $g: V(G) \rightarrow \{0,1,2,3\}$ by $g(x) = f(x)$for $x \in V(P)$,
$g(x) = 0$ for $x \in \cup_{i=1}^{k} N(u_i) \cap \big(V(G)(P)\big)$ and $g(x) = 1$
When $x \in V(G) \cup\ N(u_i)$.
Obviously g is a strong roman domination function for G and

$$\gamma_{SR}(G) \leq w(g)$$
$$w(f) + n - diam(G) - 1 - (\delta - 2)$$
$$diam(G) + 1 + n - diam(G) - 1 - (\delta - 2)$$
$$\gamma_{SR}(G)\ \leq n - (\delta - 2)$$

This completes the proof.

Theorem 5
If $G = P_4$, then $\gamma_{SR}(G) = 5$.
Proof.
G can be draw as follows

Figure 5.1 P_4
Define $f(v_1) = 0, f(v_2) = 3, f(v_3) = 0, f(v_4) = 2$.
Then f is a strong roman domination function with $f(v) = 5$.
We have to prove that f is minimal strong roman domination function.
Suppose there is a minimal strong roman domination function g such that $g < f$.

Case (1).
Let $g(v_1) = 0$, then $g(v_2) = 3$.
If $g(v_3) = 0$, then$g(v_4)$ must be $2 \vee 3$, which implies$g \geq f$, a contradiction.
If $g(v_3) = 1$, then $g(v_4) = 2$ here $g > f$, a contradiction.
If $g(v_3) = 2$, then $g(v_4) \neq 0$, now g is not minimal, a contradiction.
If $g(v_3) = 3$,then$g > f$,a contradiction.
Case (2).
Let $g(v_1) = 1$, then$g(v_2) = 2$.
If $g(v_3) = 0$, then $g(v_4) = 3$, which implies $g > f$,a contradiction.
If $g(v_3) = 1 \vee 2$, then$g(v_4) \neq 0$,here$g \geq f$,a contradiction.
If $g(v_3) = 3$, then obviously $g > f$, a contradiction.

Case (3).

Let $g(v_1) = 2$.

If $g(v_2) = 0$, then $g(v_3) = 3$, which implies $g > f$, a contradiction.

If$g(v_2) = 1$, then for any value of$g$ and $g(v_4), g > f$, a contradiction.

If $g(v_2) = 2$, then for any value of $g(v_3)$ and $g(v_4), g \geq f$,a contradiction.

If $g(v_2) = 3$, then clearly $g > f$,a contradiction.

Case (4).

Let $g(v_1) = 3$.

If $g(v_2) = 0$, and $g(v_3) = 0$, then $g(v_4) = 3$, here $g > f$,a contradiction.

If $g(v_2) = 0$and $g(v_3) = 1$, then $g(v_4) = 2$, which implies $g > f$, a contradiction.

If $g(v_2) = 0$ and $g(v_3) = 2$, then $g(v_4) \neq 0$, which implies $g > f$,a contradiction.

If $g(v_2) = 0$ and $g(v_3) = 3$, then any value of $g(v_4), g > f$, a contradiction.

If $g(v_2) = 1$, then $g(v_3) = 2$,here $g > f$,a contradiction.

If $g(v_2) = 2$, then all the value of $g(v_3)$ and $g(v_4), g > f$, a contradiction.

If$g(v_3) = 3$,clearly$g > f$, a contradiction.

Thus all the above cases, we get a contradiction.

Hence $f$is minimal strong roman domination function.

This completes the proof.

Theorem 6

For a graph G of order n with $g(G) \geq 3$ we have $\gamma_{SR}(G) \leq g(G)$.

Proof.

First note that if G is an $n-$cycle then $\gamma_{SR}(G) = n$.

Now, let C be a cycle of length $g(G)$ in G.

If $g(G) = 3 \vee 4$, then we need at least 1 or 2vertices, respectively, to dominate the vertices of C the statement follows by theorem 7.2.

Let $g(G) \geq 5$. Then a vertex not in$V(C)$, can be adjacent to at most one vertex of $C$for otherwise we obtain a cycle of length less than $g(G)$which is a contradiction. Now the result follows by theorem 7.2.

$$\gamma_{SR}(G) \geq g(G).$$

This completes the proof.

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