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ON L^1 -CONVERGENCE OF NEWLY DEFINED MODIFIED SUM

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Abstract: Many generalized and revised versions of the classes aforementioned have been introduced to study this problem of trigonometric series but assumptions on coefficients alone could not ensure the L^1-convergence of either cosine series or sine series and the condition remained confirmed. In order to have better results, Rees and Stanojevic , Kumari and Ram and many other authors start defining cosine sums suitably for studying L^1-convergence, as these sums give better results than conventional partial sums.So,motivated by the above authors we try to define the new modified sum. In order to prove this modified sum , we introduce a new class named as AW.

Keywords: New Modified Sum, AW Class, Convergence, L^1-Metric
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Introduction:

Let the cosine series be:

(1.1) $\varphi_n(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} coskx$ and $g(x)$ is denoted as its usual partial sum i.e.

$g(x) = \lim_{n\to\infty} \varphi_n(x)$

C.S.Rees , C.V.Stanojevic[3] gave the proof of the modified cosine sum stated:

$$g_n(x) = \frac{1}{2}\sum_{i=0}^{n} \Delta a_i + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \Delta a_j\right) cosix.$$

A.N. Kolmogorov[1] gives the introduction of quasi-convex sequences and also gives the proof of his result which is known as:

Theorem 2: If null sequence an is quasi-convex , then series (1.1) will converge in metric space L with the necessary and sufficient condition:

$\lim_{n\to\infty} a_n logn = 0$

C. N. Moore[2] gave the generalisation of quasi-convexity of null sequences as:,

$\sum_{n=1}^{\infty} n^k |\Delta^{k+1} a_n| < \infty,\ for\ k > 0.$

where, fractional differences are in order and proved its integrability.

Also proved by L. S. BOSANQUET[6] that if the null sequence a_n satisfying above condition; Then

$\sum_{n=1}^{\infty} n^r |\Delta^{r+1} a_n| < \infty,\ for\ k > 0\ for\ 0 \le r < k.$ And is of bounded variation.

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In this paper ,we discuss newly defined modified sum and its convergence in L^1 ,which is stated as:

(1.2.1) $\varphi_n(x) = \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(\sum_{k=m}^{n}\Delta^2(a_k))cossx$

(1.2.2) $\varphi_n(x) = \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(\sum_{k=m}^{n}\Delta^2(a_k))sinsx$

We will prove the L^1--convergence of (1.2.1) in the new class AW.

Definition: let a_n be null sequence and It belongs to the class AW if $a_n \to 0\ as\ n \to \infty$ and there will be a quasi-convex sequence a_n i.e.

$\sum_{n=1}^{\infty}(n+1)\,|\Delta^2 a_n| < \infty.$, then

$(i)\sum_{n=0}^{\infty}\delta_n < \infty.$

$(ii)\ |\Delta^2 a_n| \leq \delta_n\ , for\ all\ n.$

Now,

$$\varphi_n(x) = \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(\sum_{k=m}^{n}\Delta^2(a_k))cossx$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}\Delta(a_m)cossx - \Delta(a_{n+1})\sum_{s=1}^{n}\sum_{m=s}^{n}cossx$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(a_s - a_{n+1})cossx - \Delta(a_{n+1})\sum_{s=1}^{n}\sum_{m=s}^{n}cossx$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(a_s - a_{n+1})cossx - \Delta(a_{n+1})\,(\sum_{m=1}^{n}cosx + \sum_{m=2}^{n}cos2x + \sum_{m=3}^{n}cos3x + --------- + \sum_{m=n}^{n}cosnx$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(a_s - a_{n+1})cossx - \Delta(a_{n+1})\,[D_n(x) + \big(D_n(x) - D_1(x)\big) + \big(D_n(x) - D_2(x)\big) + ------ + \big(D_n(x) - D_{n-1}(x)\big)]$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(a_s - a_{n+1})cossx - \Delta(a_{n+1})\,[nD_n(x) - \big(D_1(x) + D_2(x) + ------ + D_{n-1}(x)\big)]$$
$$= \frac{a_0}{2} + \sum_{s=1}^{n}\sum_{m=s}^{n}(a_s - a_{n+1})cossx - \Delta(a_{n+1})\,[nD_n(x) - \sum_{S=1}^{n-1}D_s(x)].$$

As, we know the equality "$\widetilde{D}'_n(x) = (n+1)D_n(x) - (n+1)F_n(x)$" (1)

So, we have

$$= S_n(x) - a_{n+1}D_n(x) - \Delta(a_{n+1})[nD_n(x) - (n+1)F_n(x) + D_n(x)]$$
$$= S_n(x) - a_{n+1}D_n(x) - \Delta(a_{n+1})[(n+1)D_n(x) - (n+1)F_n(x)],$$

Now, By applying (1), we get,

$$= S_n(x) - a_{n+1}D_n(x) - \Delta(a_{n+1})\,\widetilde{D}'_n(x)$$

Where , $D_n(x)\ and\ \widetilde{D}'_n(x)$ respectively deonotes Drichlet kernel and Conjugate Drichlet kernel .

Now we will prove the following main result of this paper stated as:

Result: 1.3 Suppose (a_s) be a null sequence and if $(a_s)\mathcal{E}AW$ then

(a.) $\varphi_n(x) \to \varphi(x)\ pointwise;\ 0 < \delta \leq x \leq \pi.$

(b.) $\boldsymbol{\varphi}$ converges in $L(0,\pi]$.

(c.) In L[1]-metric $\varphi_n(x) \to \varphi(x)$.

Proof (a.) As in (A), it is not as much difficult as to give estimation :
$|\widetilde{D}'_n(x)| \le \frac{2\pi^2 n}{\delta^2}$
As the sequence (a_k) tends to zero . So the 1st and 2nd term goes to zero as per above estimations and for 3rd term;

$$\left|\Delta(a_{n+1})\,\widetilde{D}'_n(x)\right| \le \left|\frac{2\pi^2 n}{\delta^2}(a_{n+1})\right|$$

$= o(1)\ as\ n \to \infty$
In view of $a_s\ \mathcal{E}\ S'$
As such, we obtain that
$\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} S_n(x) = \varphi(x) \quad for\ \ 0 < x \le \pi.$

Proof (b.) $\varphi\ \mathcal{E}\ L(0,\pi]$ is an immidiate result of (1.3) as we have assumed that $a_s\ \mathcal{E}\ S'$

Proof (c.) $\varphi(x) - \boldsymbol{\varphi}_n(x) = \sum_{k=n+1}^{\infty} a_k \cos kx + a_{n+1}D_n(x) + \Delta(a_{n+1})\widetilde{D}'_n(x)$
$= \lim_{t\to\infty}(\sum_{s=n+1}^{t} a_k \sin kx)' + a_{n+1}D_n(x) + \Delta(a_{n+1})\widetilde{D}'_n(x)$
Now, we will apply summation by parts two times then the above equality reduces to:

$$\varphi(x) - \boldsymbol{\varphi}_n(x) = \lim_{s\to\infty}\Big(\sum_{s=n+1}^{t-1} \Delta(a_s)\widetilde{D}'_s(x) + a_t\widetilde{D}'_t(x) - a_{n+1}\widetilde{D}'(x)\Big) + a_{n+1}D_n(x) + \Delta(a_{n+1})\,\widetilde{D}'_n(x)$$

$= \lim_{t\to\infty}\big(\sum_{s=n+1}^{t-2}(k+1)\Delta^2(a_s)\,\tilde{F}'_s(x) + t\,\Delta(a_{t-1})\tilde{F}'_{t-1}(x) - (n+1)\Delta(s_{n+1})\tilde{F}'_n(x) + a_t\widetilde{D}'_t(x) + \Delta(s_{n+1})\widetilde{D}'_n(x)\big)$
it can be easily shown that the 2nd and 4th term in the above equality tends to zero (follows from the discussion which is made in the proof of (i)) Hence it is obtained:
$\varphi(x) - \boldsymbol{\varphi}_n(x) = \sum_{s=n+1}^{\infty}(s+1)\Delta^2(a_s)\,\tilde{F}'_s(x) - (n+1)\Delta(a_{n+1})\tilde{F}'_n(x) + \Delta(a_{n+1})\,\widetilde{D}'_n(x)$
we get,
‘’$\int_{-\pi}^{\pi}|\varphi(x) - \boldsymbol{\varphi}_n(x)|dx = \sum_{s=n+1}^{\infty}(s+1)|\Delta^2(a_s)|\int_{-\pi}^{\pi}|\tilde{F}'_s(x)|\,dx - (n+1)\Delta(a_{n+1})\int_{-\pi}^{\pi}|\tilde{F}'_n(x)|dx + \Delta(a_{n+1})\int_{-\pi}^{\pi}|\widetilde{D}'_n(x)|dx$"
As we follow the theorem “Zygmund’s” :
‘’ $\int_{-\pi}^{\pi}|\tilde{F}'_s(x)|\,dx = o(s)$"
The 1st term in the above converges as $a_n \mathcal{E}\ S'$
for the 2nd and the 3rd term ;

$$(n+1)|\Delta(a_{n+1})|\int_{-\pi}^{\pi}\left|\tilde{F}'_n(x)\right|dx = o\Big(\sum_{s=n+1}^{\infty} s^2|\Delta^2(m_s)|\Big) = o(1).$$

Hence, it follows that

$$\int_{-\pi}^{\pi} |\varphi(x) - \boldsymbol{\varphi}_n(x)| dx = o(1) \; as \; n \to \infty.$$

In the same manner we can prove the result for (1.2.2)

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