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SOME STUDIES OF THE GENERAL THEORY OF RELATIVITY'S UNIFORM AND ISOTROPIC COSMOS

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Abstract

In the framework of General Relativity and assuming the Copernican principle, accounted for by the Friedman-Lemaitre-Robertson-Walker metric, a spatially flat universe is consistent with the cosmographic conversely, this condition, which is fulfilled by astrophysical measurements, necessarily requires spatial flatness. Here, we will construct some cosmological models assuming the validity of General Relativity, of Copernican principle (homogeneity and isotropyat large scale) and on dark energy pictured either by some non-ideal fluids or by canonical scalar fields interacting with dark matter. A theorist cannot appeal to this theory in order to justify their views.

Keywords: General Theory, Relativity's, Uniform and Isotropic, Cosmos

1. Introduction

The field of cosmology is concerned with the study of the structure, history, and potential future evolution of the universe on a galactic scale, spanning billions of light-years. To investigate the physics of the cosmos, cosmologists develop theoretical models within the context of general relativity. They focus on the big picture by contrasting the seen cosmos with the models. Before the advent of general relativity, cosmologists looked on Newton's theory of gravitation to explain the cosmos. A variety of issues arose while attempting to deal with the dynamics of the cosmos using Newtonian cosmological models. The hypothesis relies on the idea that a gravitational disturbance can spread instantly, which is a controversial idea, especially when extrapolated across huge distances. This stymied the development of Newtonian theory. The general theory of relativity proposed by Einstein ultimately superseded Newtonian theory. Einstein's general theory of relativity was a major influence on modern cosmology. For the first time, this theory offers a physical and mathematical framework of general relativity to address issues of galactic proportions.[1]

2. Homogeneous and isotropic cosmological models

Einstein's general theory of relativity, which provides new ways to think about and solve issues on a cosmic scale, is often credited as the inspiration for modern cosmology. The static cosmological models used were his own creations, and they were all filled with a perfect fluid with a uniform distribution. However, the model has a number of drawbacks that make it undesirable. This runs counter to observations made by Hubble and L.Humason who found that the redshift of nebulae light increased at least very closely in a linear fashion with increasing distance. the Einstein static universe with a pressure term and gave the field equations for this situation shortly after the publication of Einstein's static model. Nothing at all, not even radiation, exists in the de-sitter cosmos. In the de-sitter universe, we learn how Hubble and Humason's measured redshift is really working. The observed

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contraction of nebulae is consistent with the de-sitter cosmos being entirely empty. The Einstein universe, on the other hand, is dense with stuff, but it fails to account for the observed receding of nebulae. Therefore, neither Einstein's nor de-sitter's universes are accurate representations of the real one. Non-static models where the metric tensor is inherently time-dependent are required to develop a model that combines the benefits of Einstein's and desitter's static models.[2]

2.1 Standard Model and Cosmological Constant

Friedmann used the Cosmological Principle to solve Einstein's field equations, and the resulting non-static cosmological solutions are consistent with an expanding universe. Therefore, the Friedmann-Robertson-Walker (FRW) metric is the best line element for representing a non-static and homogenous model of the cosmos.[3] In standard spherical coordinates $(x_i) = (t, r, \theta, \phi)$, a spatially homogeneous and isotropic FRW line element has the form (in units $c = 1$)

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\emptyset^2)\right] \qquad (2.1)$$

Where, $a(t)$is the cosmic scale factor that describes the expansion or contraction of the universe in terms of distances (scales), and is related to the red shift of the 3-space; k is the curvature parameter that describes the geometry of the spatial section of space-time, with closed, flat, and open universes corresponding to k = -1, 0, and 1, respectively. Amazingly, the FRW models have been able to satisfactorily describe the observed characteristics of the cosmos.[4]

The Einstein's field equations (2.1), for the metric (2.2), in case of the energy momentum tensor, reduce to the following equations:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho - \frac{k}{a^2} \qquad (2.2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \qquad (2.3)$$

Where, an over dot denotes derivative with respect to the cosmic time t.

For the FRW space-time (2.1) and the perfect fluid energy-momentum tensor, yields a single conservation equation

$$\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0 \qquad (2.4)$$

In reality, this equation cannot exist without the Friedmann equations. This means that variationsin the energy density at a given location are possible in the cosmos (as defined by the Hubble parameter $H = \frac{\dot{a}}{a}$). Due to the free flow of energy between matter and the space-time geometry,it is important to keep in mind that the concept of "total energy" does not hold.[5]

The Friedmann-Robertson-Walker models are fundamental in the study of the cosmos. These models may not be perfect representations of the cosmos, but they do provide usefulglobal approximations of the universe as it is now. All directions from a given place in space are considered to be equal (isotropy) and (i) the cosmos is the same everywhere (spatial homogeneity).

Anisotropic cosmological models have received renewed theoretical interest in recent years thanks to experiments measuring cosmic microwave radiation and speculations about the quantity of helium generated in the early phases of the universe. Bianchi models, which are both Spatially Homogenous and anisotropic, play a significant role in contemporary cosmology because they bridge the gap between FRW models and a wholly inhomogeneous and anisotropic cosmos. Since three-dimensional groups function simply transitively on space in the same way that three-dimensional orbits do, they are required in every spatially homogeneous Bianchimodel. It's helpful to assume that the cosmos is homogeneous and isotropic, meaning that all directions in space are equal and no region is fundamentally different from any other. There is some circumstantial evidence that the distribution of these materials on vast scales exhibit isotropy, and this includes the distribution of galaxies in the sky along their apparent magnitudes and red shifts, the distribution of radio sources, the cosmic X-ray background, and the cosmic microwave background.[6]

2.2 Spatially Homogeneous and Anisotropic Models

There are no observational facts that ensure in an era previous to the recombination that the cosmos was isotropic and homogenous at the time. It's not known what kinds of matter fields existed in the early universe. Assumptions of spherical symmetry and isotropy are not strictly valid near the Big-Bang singularity, therefore a smoothed-out depiction of the early cosmos are impossible. In order to understand the origins of modern-day local anisotropies in galaxies, clusters, and super clusters, studying anisotropy at early times is a natural next step. Several

probable causes have been proposed for these anisotropies. These include cosmic magnetic or electric fields, long-wavelength gravitational waves, Yang-Mills fields, and others. In addition, theoretical interest in the cosmological models with anisotropic background has been sparked by experimental studies of the isotropy of the CMBR and speculation about the amount of helium generated in the early stages of the evolution of the universe. Therefore, it seems reasonable to assume a geometry that is more generic than merely the isotropy and homogeneous FRW geometry in order to characterise the early evolution of the cosmos. Understanding the early behaviour of the universe is crucial, and anisotropic cosmological models play a crucial part in this. Understanding the evolution of the universe and the factors that will shape its future are primary goals of modern cosmology.[7]

3. The Friedmann model

The cosmological principle, the absence of long-range antigravity, and the assumption that no additional matter or energy is created after the Big-Bang leave just three alternatives open, according to the general theory of relativity. In recognition of Alexander Friedmann, who first derived them analytically in the 1920s these are commonly referred to as Friedmann worlds. The pace of expansion slows with time in every universe, but each universe has a different eventual fate that depends on the average density of its matter in comparison to a critical density.[8]

If we define the average matter density, divided by the critical density to be Ω_M (the subscript M stands for matter),

$$\Omega\mathrm{M} = \text{Avergae density of}\,\frac{\text{matter}}{\text{Cririal Density}}$$

Then, the three possible universes correspond to the cases are given blow

ΩM > 1.

ΩM = 1.

ΩM < 1.

If ΩM > 1, it means, there is an excess of average density over the critical density. In this universe, the recession speed grows negative as time progresses, and galaxies finally reverse course and begin to approach one another. The Big Crunch is the ultimate fate of this cosmos.[9]

If ΩM = 1, It means, the critical density is coincident with the mean density. In this universe, the pace at which galaxies recede from one another decreases with time, eventually approaching zero as time approaches infinity. So, the universe will keep growing forever. Most inflationary cosmologies predict this kind of universe.

If ΩM < 1, It means, In general, there isn't enough matter to reach the critical density. With moreand more time passing, the recession speed (for a particular pair of galaxies) in this universe approaches a constant, nonzero value. So, it's no sweat for the Universe to keep growingindefinitely.[10]

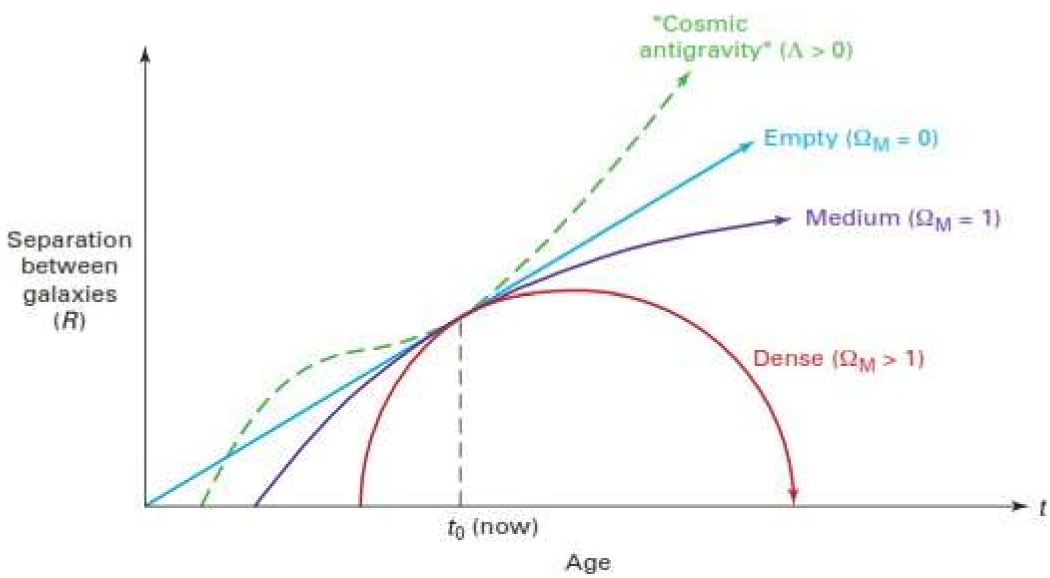

Figure 3.1: Big-Bang models of the Universe are shown; the vertical axis represents the separation between any two galaxies (preferably in different super clusters), and the horizontal axis is time

Each of these three universe types has its own unique geometry. Flat universes, or (critical universes), are the term used to describe the condition where M = 1. The geometry developed by the Greek mathematician Euclid in the third century b.c., known as "Euclidean geometry," is used to describe this. The "fifth postulate" of Euclid states that if you have a line and a point that is not on the line, then you can only draw one parallel line across the point (Fig.(1.5)-A). A cosmos like this has no depth, no width, and no height, yet it hardly stretches eternally. It's age is

exactly two-thirds of the Hubble time, i.e. $\frac{2}{3}/H_0 = (2/3)\ T_0$.

For ΩM > 1 universe, the fifth postulate of Euclid is invalid. No two parallel lines can be drawn via a location that is not on a line (Fig. (3.2)-B). The space-time of such a universe is positively curved. The size of the universe is bounded but not infinite. A hot "big crunch" is where it's at now. Another term for a closed universe is a "positively curved" one. Age estimates put it as less than 0.6 c.[11]

For ΩM < 1 universe, the fifth postulate of Euclid is invalid. Given a line and a point off of it, it is possible to create an endless number of parallel lines through the off-line point (Fig.(3.2)- C). A universe with negative spatial curvature is readily unlimited in size since its volume is open and infinite. An open universe, hyperbolic universe, or negatively curved universe are all names for the same thing. Its age is between (2/3) T_0 and T_0 (if ΩM = 0, the age is T0). ΩM = 1 that is flat universe, represents the dividing line between open universe and closed universe., which corresponds to a flat universe.[12]

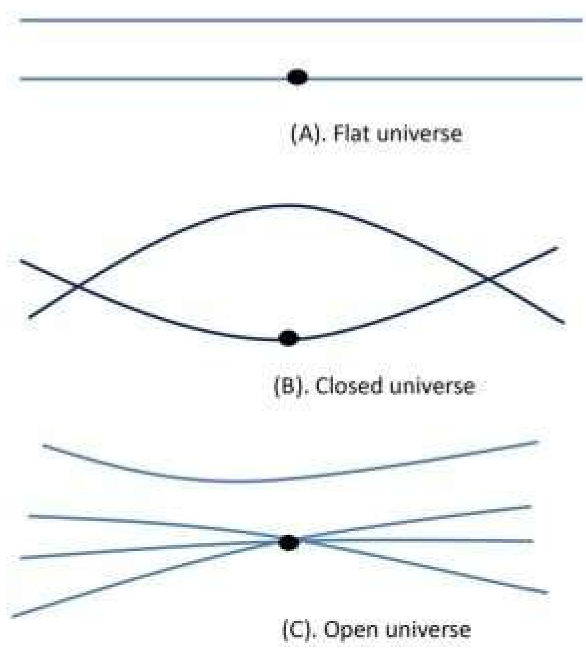

Figure 3.2: Three kinds of universes

4 The physical and geometrical properties

The average Hubble parameter and the direction-specific Hubble parameters are:

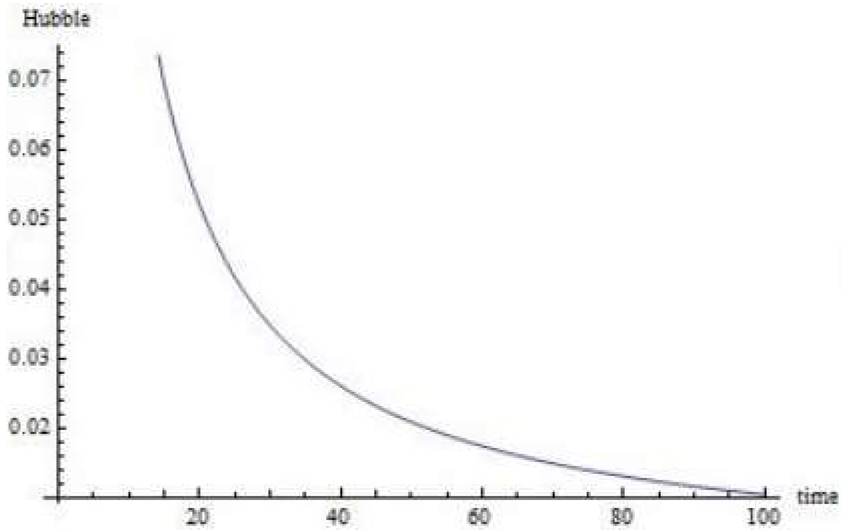

Figure 4.1: Hubble's parameter (H) versus cosmic time (t).

Here $k_5 = 1$, $\alpha = 0.2$ and $k_3 = 1$

$$H_1 = H_2 = \frac{k_5 k_3}{(\alpha+1)(k_5 t + k_6)} \qquad (4.1)$$

$$H_3 = \frac{k_5}{(\alpha+1)(k_5 t + k_6)} \qquad (4.2)$$

$$H_4 = \frac{k_5(k_3+1)}{(\alpha+1)(k_5 t + k_6)} \qquad (4.3)$$

$$H = \frac{k_5(3k_3+2)}{4(\alpha+1)(k_5 t + k_6)} \qquad (4.4)$$

$$\theta = 4H = \frac{k_5(3k_3+2)}{(\alpha+1)(k_5 t + k_6)} \qquad (4.5)$$

With this model, the shear scalar is

$$\sigma^2 = \frac{{k_5}^2\left(\frac{3}{4}{k_3}^2 - k_3 + 1\right)}{2((\alpha+1)\left(k_5 t + k_6)\right)^2} \qquad (4.6)$$

Model anisotropic sensitivity

$$\bar{A} = \frac{3k_3{}^2 - 4k_3 + 16}{(3k_3 t + 2)^2} \tag{4.7}$$

The energy density of the model is

$$8\pi\epsilon = k_5{}^2\left(3k_3{}^2 + 5k_3 + 1\right)\left((\alpha + 1)(k_5 t + k_6)\right)^{-2} + \Lambda \tag{4.8}$$

Model pressure has increased

$$8\pi p = k_5{}^2[(\alpha + 1)(k_5 t + k_6)]^{-2}\left[-3k_3{}^2 + 2\alpha(k_3 + 1) - 3K_3 - 1 - \frac{32}{3}\pi l(3k_3 + 2)\right] + 8\pi k_5\zeta(3k_3 + 2)[(\alpha + 1)(k_5 t + k_6)]^{-1} - \Lambda \tag{4.9}$$

Ellis specifies the following energy parameters $(i)(\epsilon + p) > 0$ and $(ii)(\epsilon + 3p) > 0$ [13]

The result of (i)'s energy state is

$$k_5{}^2[(\alpha + 1)(k_5 t + k_6)]^{-2}\left[2k_3 + 2\alpha(k_3 + 1) - \frac{32}{3}\pi l(3k_3 + 2)\right] + 8\pi k_5\zeta(3k_3 + 2)[(\alpha + 1)(k_5 t + k_6)]^{-1} > 0 \tag{4.10}$$

as a result of (ii)'s energy constraints

$$k_5{}^2[(\alpha + 1)(k_5 t + k_6)]^{-2}\left[-6k_3{}^2 - 4k_3{}^2 + 6\alpha(k_3 + 1) - 32\pi l(3k_3 + 2)\right] + 24\pi k_5\zeta(3k_3 + 2)[(\alpha + 1)(k_5 t + k_6)]^{-1} - 2\Lambda > 0 \tag{4.11}$$

We get the following (4.11) from the previous equation:

$$k_5{}^2[(\alpha + 1)(k_5 t + k_6)]^{-2}\left[-6k_3{}^2 - 4k_3{}^2 + 6\alpha(k_3 + 1) - 32\pi l(3k_3 + 2)\right] + 24\pi k_5\zeta(3k_3 + 2)[(\alpha + 1)(k_5 t + k_6)]^{-1} > 2\Lambda \tag{4.12}$$

4.1 The special model

The model is simplified when the appropriate coordinates and constants are used (specifically, when $k_5 = 1$ and $k_6 = 0$):

$$ds^2 = -dt^2 + m^2 K_4{}^2[(\alpha + 1)(t)]^{\left(\frac{2k_3}{\alpha+1}\right)}dx^2 + K_4{}^2[(\alpha + 1)(t)]^{\left(\frac{2k_3}{\alpha+1}\right)}dy^2 + [(\alpha + 1)(t)]^{\left(\frac{2}{\alpha+1}\right)}dz^2 + K_4{}^2[(\alpha + 1)(t)]^{\left(\frac{2(k_3+1)}{\alpha+1}\right)}\mathrm{du}^2 \tag{4.13}$$

The values of the Hubble expansion parameter () and the shear () are determined by:

$$H = \frac{(3k_3 + 2)}{4(\alpha + 1)(t)} \tag{4.14}$$

$$\theta = 4H = \frac{(3k_3 + 2)}{(\alpha + 1)(t)} \tag{4.15}$$

$$\sigma^2 = \frac{\left(\frac{3}{4}k_3{}^2 - k_3 + 1\right)}{2\left((\alpha + 1)t\right)^2} \tag{4.16}$$

The formula for the anisotropic parameter is:

$$\bar{A} = \frac{3{k_3}^2 - 4k_3 + 16}{(3k_3 t + 2)^2} \qquad (4.17)$$

Density and pressure expressions for the model are as follows (4.13) [14]

$$8\pi\epsilon = \left(3{k_3}^2 + 5k_3 + 1\right)\left((\alpha + 1)t\right)^{-2} + \Lambda \qquad (4.18)$$

And

$$8\pi p = [(\alpha + 1)(t)]^{-2}\left[-3{k_3}^2 + 2\alpha(k_3 + 1) - 3K_3 - 1 - \frac{32}{3}\pi l(3k_3 + 2)\right] + 8\pi\zeta(3k_3 + 2)[(\alpha + 1)(t)]^{-1} - \Lambda \qquad (4.19)$$

The current energy situation is $(i)(\epsilon + p) > 0\ and\ (ii)(\epsilon + 3p) > 0$

The result of (i)'s energy state is

$$[(\alpha + 1)(t)]^{-2}\left[2k_3 + 2\alpha(k_3 + 1) - \frac{32}{3}\pi l(3k_3 + 2)\right] + 8\pi\zeta(3k_3 + 2)[(\alpha + 1)(t)]^{-1} > 0 \ (4.20)$$

as a result of (ii)'s energy constraints

$$[(\alpha + 1)(t)]^{-2}\left[-6{k_3}^2 - 4{k_3}^2 + 6\alpha(k_3 + 1) - 32\pi l(3k_3 + 2)\right] + 24\pi\zeta(3k_3 + 2)[(\alpha + 1)(t)]^{-1} > 2\Lambda \qquad (4.21)$$

5. Cosmological Constant

Because of its unusual features, the cosmological constant has a long history of support and opposition. Since then, gravitational theory has been given a jolt by the cosmological constant. After Hubble's data confirmed the expanding universe picture, Einstein conceded that the -term in his gravitational field equations was superfluous. He went so far as to call the adoption of the Λ-term his "biggest blunder." However, prominent cosmologists of the, like as Lemaitre and Eddington, felt that models based on the -term introduced has certain compelling aspects into cosmology and that these should also be studied at length. A new understanding developed when interpreted the -term as a Lorentz-invariant vacuum 'fluid' in Einstein's equation. In a very novel move, Zel'dovich resurrected the question of the cosmological constant by linking it to the vacuum energy density resulting from quantum fluctuations. So, with a fresh perspective, people began considering again. The cosmological constant was gaining theoretical traction in this way, allowing it to endure over time. There was no astronomical evidence for before 1998, and the observational upper bound was so large (10121 Planck units) that many particle physicists assumed it must have a value of zero by some basic principle.[15]

5. Conclusion

The concept you're referring to is likely the "cosmological principle," which is a fundamental assumption in cosmology that states the universe is homogeneous (uniform) and isotropic (looks the same in all directions) on large scales. the cosmological principle have led to the understanding that the universe on large scales is homogeneous and isotropic. This understanding forms the foundation of modern cosmology and has provided insights into the universe's history, structure, and evolution.

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