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ON A SEMI-PERIODIC BOUNDARY VALUE PROBLEM FOR THE THREE-DIMENSIONAL TRICOMI EQUATION IN AN UNBOUNDED PARALLELEPIPED

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Abstract. The unique solvability of a generalized solution of one semi-periodic boundary value problem for the three-dimensional Tricomi equation in an unbounded parallelepiped is studied in the article by the methods of “ε −regularization” and a priori estimates using the Fourier transform.

Keywords: three-dimensional Tricomi equation, semi-periodic boundary value problem, problem correctness, “ε −regularization” method, a priori estimates, Fourier transform.

1. Introduction and problem statement

A.V. Bitsadze showed in [1] that the Dirichlet problem is incorrect for an equation of mixed type [1]. The question arises: is it possible to replace the conditions of the Dirichlet problem with other conditions covering the entire boundary, which ensure the correctness of the problem? Such boundary value problems (nonlocal boundary value problems) for an equation of mixed type were proposed and studied for the first time by F.I. Frankl [2]. Problems for a mixed-type equation of the first kind in bounded domains, close in formulation to the ones under study, were investigated in [3, 4].

In this paper, using the results given in [3-10], we study the unique solvability and smoothness of the generalized solution to a semi-periodic boundary value problem in an unbounded domain.

In domain

$$G = (-1,1)\times(0,T)\times R = Q\times R = \{(x,t,z),\ x\in(-1,1),\ 0<t<T<+\infty, z\in R\},$$

we consider the Tricomi equations:

$$Lu = xu_{tt} - \Delta u + a(x)u_t + c(x,t)u = f(x,t,z), \tag{1}$$

where $\Delta u = u_{xx} + u_{zz}$ is the Laplace operator.

Let all coefficients of equation (1) be sufficiently smooth functions in domain Q.

To solve the problems posed, we need to introduce definitions of several function spaces and notation.

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Let the Fourier transform be denoted by $\hat{u}(x,t,\lambda)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}u(x,t,z)e^{-i\lambda z}dz$, of function $u(x,t,z)$, and the inverse Fourier transform be denoted by $u(x,t,z)=(2\pi)^{-1/2}\int_{-\infty}^{+\infty}\hat{u}(x,t,\lambda)e^{i\lambda z}dz$. Now, using the Fourier transform, we define space $W_2^{l,s}(G)$ with norm

$$\|u\|^2_{W_2^{l,s}(G)}=(2\pi)^{-1/2}\cdot\int_{-\infty}^{+\infty}(1+|\lambda|^2)^s\cdot\|\hat{u}(x,t,\lambda)\|^2_{W_2^l(Q)}d\lambda, \tag{A}$$

where $s,l-$ are any finite positive integers, and the norm in the Sobolev space $W_2^l(Q)$, is defined as follows

$\|\vartheta\|^2_{W_2^l(Q)}=\sum_{|\alpha|\le l}\int_Q|D^\alpha\vartheta|^2dxdt$, $\alpha-$ is a multi-index, $D^\alpha-$ is a generalized derivative with respect to variables x and t. Obviously, space $W_2^{l,s}(G)$ with norm (A) is a Banach space [6,11].

Semi-periodic boundary value problem.

The task is to find a generalized solution $u(x,t,z)$ to equation (1) from space $W_2^{2,3}(G)$, satisfying the following boundary conditions

$$D_t^p u\big|_{t=0}=D_t^p u\big|_{t=T}, \tag{2}$$

$$u\big|_{x=-1}=u\big|_{x=1}=0 \tag{3}$$

for $p=0,1$, where $D_t^p u=\dfrac{\partial^p u}{\partial t^p}$, $D_t^0 u=u$.

Furthermore, we assume that $u(x,t,z)$ and $u_z(x,t,z)\to 0$ if $|z|\to\infty$, $u(x,t,z)$ is absolutely integrable over z on R for any (x,t) in $\overline{Q}$ (4)

Definition 1. A generalized solution to problem (1)-(4) is function $u(x,t,z)\in W_2^{2,2}(G)$ that satisfies equation (1) almost everywhere in domain G, under conditions (2)-(4).

Theorem 1. (Main result). Let the following conditions be satisfied for the coefficients of equation (1); $2a(x)+\mu x\ge\delta_1>0$, $\mu c(x,t)-c_t(x,t)\ge\delta_2>0$, for all $(x,t)\in\overline{Q}$, where

$\mu - const > 0, \quad c(x,0) = c(x,T),$ for all $x \in [-1,1].$ Then for any function $f \in W_2^{1,3}(G)$ such that $f(x,0,z) = f(x,T,z),$ there exists a unique generalized solution to problem (1) - (4) from space $W_2^{2,3}(G).$

For the solution of problem (1) - (4) the following estimates are valid:

I) $\|u\|^2_{W_2^{1,3}(G)} \le c_1 \|f\|^2_{W_2^{0,3}(G)}$

II) $\|u\|^2_{W_2^{2,3}(G)} \le c_2 \|f\|^2_{W_2^{1,3}(G)}$

in what follows, we denote positive, generally speaking, different constant non-zero numbers by c_i

The unique solvability of problem (1) - (4) is proved using the Fourier transform.

Applying the Fourier transform for problem (1) - (4), we obtain the following problem in domain $Q = (-1,1) \times (0,T)$

$$L\hat{u} = x\hat{u}_{tt} - \hat{u}_{xx} + a(x)\hat{u}_t + (c(x,t) + \lambda^2)\hat{u} = \hat{f}(x,t,\lambda), \tag{5}$$

$$D_t^p \hat{u}\big|_{t=0} = D_t^p \hat{u}\big|_{t=T}, p = 0,1 \tag{6}$$

$$\hat{u}\big|_{x=-1} = \hat{u}\big|_{x=1} = 0, \tag{7}$$

where $\lambda \in R = (-\infty, \infty),$

$$\hat{f}(x,t,\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} f(x,t,z) e^{-i\lambda z} dz$$

Fourier transform in variable $z,$ of function $f(x,t,z).$

The unique solvability and smoothness of problem (5) - (7) was studied in [3, 4, 5, 6]. We will briefly present these results.

2. Uniqueness of the solution to problem (5) - (7).

Theorem 2. Let the following conditions be satisfied for the coefficients of equation (5); $2a(x) - \mu x \ge \delta_1 > 0,$ $\mu\, c(x,t) - c_t(x,t) \ge \delta_2 > 0,$ where $\mu - const > 0,$ for all $(x,t) \in \overline{Q},$ $c\,(x,0) = c(x,T),$ for all $x \in [-1,1].$ Then, if for any function $\hat{f}(x,t,\lambda) \in L_2(Q)$ there exists a generalized solution to problem (5) - (7) from space $W_2^2(G),$ then it is unique.

Proof. Let us prove the uniqueness of the solution to problem (5) - (7) using the energy integral method. Let there exist a solution to problem (5) - (7) from $W_2^2(Q)$. Consider the following identity

$$(L\hat{u}, 2\hat{u}_t + \mu\hat{u})_0 = (\hat{f}, 2\hat{u}_t + \mu\hat{u})_0, \tag{8}$$

where $\mu = const > 0$. By the conditions of Theorem 2, integrating identity (8) by parts, using Cauchy's inequalities with σ [12], it is easy to obtain the following necessary first estimate

$$\|\hat{u}\|^2_{W_2^1(Q)} \le c_1 \left\|\hat{f}\right\|^2_{L_2(Q)}. \tag{9}$$

Here, the uniqueness of the solution to problem (5) - (7) follows from $W_2^2(Q)$. ***Theorem 2 is proved.***

3. Third-order equation with a small parameter

The solvability of problem (5) - (7) is proved by the "ε-regularization" method, namely, in domain $Q = (-1,1) \times (0,T)$ we consider a family of third-order equations with a small parameter

$$L_\varepsilon \hat{u}_\varepsilon = -\varepsilon \frac{\partial^3 \hat{u}_\varepsilon}{\partial t^3} + L\hat{u}_\varepsilon = \hat{f}(x,t,\lambda) \tag{10}$$

and the following semi-periodic boundary conditions

$$D_t^q \hat{u}_\varepsilon\big|_{t=0} = D_t^q \hat{u}_\varepsilon\big|_{t=T}; q = 0,1,2, \tag{11}$$

$$\hat{u}_\varepsilon\big|_{x=-1} = \hat{u}_\varepsilon\big|_{x=1} = 0 \tag{12}$$

where $\varepsilon -$ is the small positive number, $D_z^q w = \dfrac{\partial^q w}{\partial z^q}$, $q = 1,2$; $D_z^0 w = w$.

Below we use systems of third-order equations with a small parameter (10) as a "ε-regularizing" equation for the Tricomi equation (5) [4-10].

Let us define spaces of functions $W(Q) = \{\hat{u}_\varepsilon | ; \hat{u}_\varepsilon \in W_2^2(Q), \hat{u}_{\varepsilon ttt} \in L_2(Q)\}$ satisfying the corresponding conditions (11), (12) with finite norm

$$\left\|\left|\hat{u}_\varepsilon\right|\right\|^2_W = \varepsilon \left\|\hat{u}_{\varepsilon ttt}\right\|^2_0 + \left\|\hat{u}_\varepsilon\right\|^2_2. \tag{B}$$

It is obvious, that space $W(Q)$ with norm (B) is a Banach space [11,12].

Definition 2. A generalized solution to problem (10) - (12) is a function $\{\hat{u}_\varepsilon(x,t,\lambda)\} \in W(Q)$ that satisfies equation (10) almost everywhere in domain Q under conditions (11), (12).

Theorem 3. Let the following conditions be satisfied for the coefficients of equation (10); besides let $2a(x)-\mu x\geq \delta_1>0,\ \mu\, c(x,t)-c_t(x,t)\geq \delta_2>0$, where $\mu-const>0$, for all $(x,t)\in\overline{Q},\ c(x,0)=c(x,T)$, for all $x\in[-1,1]$. Then, for any function $\hat{f}(x,t,\lambda)\in L_2(Q)$ such that $\hat{f}(x,0,\lambda)=\hat{f}(x,T,\lambda)$ there exists a generalized solution to problem (10) - (11) from space $W(Q)$ and the following estimates hold

III) $\quad \varepsilon\|\hat{u}_{\varepsilon tt}\|_0^2+\|\hat{u}_\varepsilon\|_1^2\leq c_1\left\|\hat{f}\right\|_0^2,$

IV) $\quad \varepsilon\|\hat{u}_{\varepsilon ttt}\|_0^2+\|\hat{u}_\varepsilon\|_2^2\leq c_2\left\|\hat{f}\right\|_1^2,$

Proof. The proof of Theorem 3 is realized in stages, using the Galerkin method, obtaining the corresponding a priori estimates [6].

First, we prove $III)-$ the third estimate.

Consider the identity:

$$\int_Q L_\varepsilon\hat{u}_\varepsilon\cdot(2\hat{u}_{\varepsilon t}+\mu\hat{u}_\varepsilon)\,dx\,dt=\int_Q \hat{f}\cdot(2\hat{u}_{\varepsilon t}+\mu\hat{u}_\varepsilon)\,dx\,dt. \tag{13}$$

Integrating identity (13) by parts, taking into account the condition of Theorem 3, it is easy to obtain III) – the third a priori estimate, similar to estimate (9), whence the uniqueness of the generalized solution of problem (10) - (12) follows.

Now we prove $IV)-$ the fourth estimate.

To do this, consider the identity:

$$-2\int_Q e^{-\mu t}\cdot L_\varepsilon\hat{u}_\varepsilon\cdot P\hat{u}_\varepsilon\,dx\,dt=-2\int_Q e^{-\mu t}\cdot\hat{f}\cdot P\hat{u}_\varepsilon\,dx\,dt. \tag{14}$$

where $P\hat{u}_\varepsilon=(\hat{u}_{\varepsilon ttt}-\mu\hat{u}_{\varepsilon tt}+\frac{\mu}{2}\hat{u}_{\varepsilon xx}-\mu\hat{u}_{\varepsilon t})$.

Integrating (14) by parts, taking into account the conditions of Theorem 3 and boundary conditions (11), (12), we obtain the necessary fourth estimate:

$$\varepsilon\|\hat{u}_{\varepsilon ttt}\|_0^2+\|\hat{u}_\varepsilon\|_2^2\leq c_2\left\|\hat{f}\right\|_1^2 \tag{15}$$

From the estimates proved, we obtain the unique solvability of problem (10) - (12) in space $W(Q)$. ***Theorem 3 is proved.***

Let us proceed to the proof of the solvability of problem (5) - (7).

Theorem 4. Let all conditions of Theorems 2.3 be satisfied. Then a generalized solution to problem (5) - (7) exists and is unique in $W_2^2(Q)$.

Proof. The uniqueness of the solution to problem (5) - (7) in space $W_2^2(Q)$ was proved in Theorem 2. Now we prove the existence of a solution to problem (5) - (7) in $W(Q)$. To do this, consider equation (10) and boundary conditions (11), (12) in domain Q for $\varepsilon > 0$. Since all the conditions of Theorem 3 are satisfied, there is a unique generalized solution to problem (10) - (12) in $W(Q)$, for $\varepsilon > 0$, and the third and fourth estimates are valid for it. From this it follows, by the known compactness theorem [12], that from the set of functions $\{\hat{u}_{\varepsilon}(x,t,\lambda)\}, \varepsilon > 0$, a weakly convergent subsequence of functions can be extracted, such that $\{\hat{u}_{\varepsilon_i}(x,t,\lambda)\} \to \hat{u}(x,t,\lambda)$ as $\varepsilon_i \to 0$ in $W(Q)$. Let us show that the limit function $\hat{u}(x,t,\lambda)$ satisfies equation $L\hat{u} = \hat{f}$ (equation (5)) almost everywhere in $W_2^2(Q)$. Indeed, since subsequence $\{\hat{u}_{\varepsilon_i}(x,t,\lambda)\}$ converges weakly in $W(Q)$, and subsequence $\{\sqrt{\varepsilon_i}\hat{u}_{\varepsilon_i ttt}(x,t,\lambda)\}$ is uniformly bounded in $L_2(Q)$ and operator L is linear, we have

$$L\hat{u} - \hat{f} = L\hat{u} - L\hat{u}_{\varepsilon_i} + \varepsilon_i \frac{\partial^3 \hat{u}_{\varepsilon_i}}{\partial t^3} = L(\hat{u} - \hat{u}_{\varepsilon_i}) + \varepsilon_i \frac{\partial^3 \hat{u}_{\varepsilon_i}}{\partial t^3}. \tag{16}$$

From equality (16), passing to the limit as $\varepsilon_i \to 0$, we obtain a unique generalized solution to problem (5)-(7) from the Sobolev space $W_2^2(Q)$ [6, 13]. Thus, ***Theorem 4 is proved.***

4. Existence of a solution to problem (1) - (4).

We now turn to the proof of Theorem 1, on the unique solvability of the generalized solution to problem (1) - (4) from space $W_2^{2,3}(G)$.

Proof of Theorem 1. In Theorem 2, the validity of estimate (9) is proved for the solution to problem (5) - (7), that is, the following holds

$$\|\hat{u}\|^2_{W_2^1(Q)} \le c_1 \left\|\hat{f}\right\|^2_{L_2(Q)}.$$

Multiplying inequality (8) by $(2\pi)^{-1/2} \cdot (1+|\lambda|^2)^3$ and integrating over λ from $-\infty$ to $+\infty$, we obtain

$$\|u\|^2_{W_2^{1,3}(G)} = (2\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} (1+|\lambda|^2)^3 \cdot \|\hat{u}\|^2_{W_2^1(Q)}\, d\lambda \le$$

$$\le (2\pi)^{-1/2} \cdot c_1 \cdot \int_{-\infty}^{+\infty} (1+|\lambda|^2)^3 \cdot \left\|\hat{f}\right\|^2_{L_2(Q)} d\lambda = c_1 \|f\|^2_{W_2^{0,3}(G)} \tag{17}$$

which implies the fulfillment of the first a priori estimate of Theorem 1 and the uniqueness of the generalized solution of problem (1) - (4) from $W_2^{2,3}(G)$.

Using the condition of Theorems 3, 4 with the passage to the limit as $\varepsilon \to 0$ in the fourth estimate, it is easy to obtain the following estimates for the solution to problem (1) - (4)

$$\|\hat{u}\|^2_{W_2^2(Q)} \le c_2 \left\|\hat{f}\right\|^2_{W_2^1(Q)} \tag{18}$$

Multiplying inequality (18) by $(2\pi)^{-1/2} \cdot (1+|\lambda|^2)^3$ and integrating over λ from $-\infty$ to $+\infty$, we obtain

$$\|u\|^2_{W_2^{2,3}(G)} = (2\pi)^{-1/2} \cdot \int\limits_{-\infty}^{+\infty} (1+|\lambda|^2)^3 \|\hat{u}\|^2_{W_2^2(Q)} d\lambda \le$$

$$\le (2\pi)^{-1/2} \cdot c_2 \cdot \int\limits_{-\infty}^{+\infty} (1+|\lambda|^2)^3 \cdot \left\|\hat{f}\right\|^2_{W_2^1(Q)} d\lambda = c_2 \|f\|^2_{W_2^{1,3}(G)} \tag{19}$$

which implies the validity of the second estimate of Theorem 1 and the existence of a generalized solution to problem (1) - (4) from space $W_2^{2,3}(Q)$. ***Theorem 1 is proved.***

5. Smoothness of the generalized solution to problem (1)-(3).

Now let us turn to the study of the smoothness of the generalized solution to problem (1)-(3) in spaces $W_2^{m+2,s}(Q)$, where $s,m-$ are finite integers such that

m$\ge$0, s$\ge$3.

Below, for simplicity, we assume that the coefficients of equation (1) are sufficiently differentiable functions in closed domain $\overline{Q}$.

Theorem 5. Let all the conditions of Theorem 1 be satisfied, in addition, let $D_t^q c\big|_{t=0} = D_t^q c\big|_{t=T}$. Then for any function $f \in W_2^{m+1,s}(Q)$, such that $D_t^q f\big|_{t=0} = D_t^q f\big|_{t=T}$ there exists (and, at that, the only one) generalized solution to problem (1)-(4) from space $W_2^{m+2,s}(Q)$, where $s,m-$ are any finite positive integers such that $s \ge m+3, \quad m = 0,1,2,3,...$

Proof. We note that in [3], for a mixed-type second-order equation of the first kind (5), the smoothness of the generalized solution of the nonlocal boundary value problem (5) – (7) in Sobolev spaces $W_2^{m+2,s}(Q)$ was studied and the corresponding estimates were proved

$$\|\hat{u}\|^2_{W_2^{m+2}(Q)} \le c_{m+1} \left\|\hat{f}\right\|^2_{W_2^{m+1}(Q)} \ (m = 0,1,2,3,4,...). \tag{21}$$

To prove $D_z^{s-1}u \in L_2(G)$, where $s \geq m+3$, $m = 0,1,2,3,...$, and to apply the Sobolev embedding theorem, we need to multiply inequality (21) by $(2\pi)^{-1/2} \cdot (1+|\lambda|^2)^s$ and integrate it over λ from $-\infty$ to $+\infty$, then, we obtain

$$\|u\|^2_{W_2^{m+2,s}(G)} = (2\pi)^{-1/2} \cdot \int_{-\infty}^{+\infty} (1+|\lambda|^2)^s \cdot \|\hat{u}\|^2_{W_2^{m+2}(Q)} d\lambda \leq$$

$$\leq (2\pi)^{-1/2} \cdot c_{m+1} \cdot \int_{-\infty}^{+\infty} (1+|\lambda|^2)^s \cdot \left\|\hat{f}\right\|^2_{W_2^{m+1}(Q)} d\lambda = c_1 \|f\|^2_{{}_2^{m+1,s}(G)} \quad (22)$$

Hence we obtained the existence of a unique generalized solution to problem (1)-(4) from space $W_2^{m+2,s}(G)$. ***Theorem 5 is proved.***

References

1. A.V. Bitsadze. Ill-posedness of the Dirichlet problem for equations of mixed type // DAN SSSR, 1953, Vol. 122, No. 2, p. 167-170.

2. Frankl F.I. Subsonic flow past profile with a supersonic zone ending in a direct shock wave. // Applied Mathematics and Mechanics. 1956, V.20, No. 2, p.196-202.

3. Kalmenov T.Sh. On a semi-periodic problem for a multidimensional equation of mixed type // Differential Equations. 1978. V. 14, No. 3, p. 546-548.

4. Dzhamalov S.Z., Ashurov R.R. On the smoothness of a nonlocal boundary value problem for the multidimensional Chaplygin equation in space. // Kazakh mathematics journal. - 2018, V.18, No. 2, p. 59-70.

5 . Dzhamalov S.Z., Ashurov R.R., Turakulov Kh. Sh. On the semi-nonlocal boundary value problem for three-dimensional Tricomi equation in unbounded prismatic domain// “Bulletin KRASES” Phys.-Math. Sci. 2021, V. 35 No. 2, p. 8-16.

6. Dzhamalov S.Z. Nonlocal boundary and inverse problems for equations of mixed type. // Monograph. Tashkent. 2021, 176 p.

7. Dzhamalov S.Z., Ashurov R.R.,Turakulov Kh. Sh. On a nonlocal boundary value problem of periodic type for the three-dimensional Tricomi equation in an unbounded prismatic domain // “Bulletin of the Institute of Mathematics” 2021, Vol. 4, №3, p.52-59.

8. Turakulov Kh. Sh. On a nonlocal boundary value problem of poly periodic type for the three-dimensional Tricomi equation in an unbounded parallelepiped domain // “Bulletin of the Institute of Mathematics” 2022, Vol. 5, №1, p.1-6.

9 . Dzhamalov S.Z., Ashurov R.R., Turakulov Kh. Sh., The linear inverse problem for the three-dimensional Tricomi equation in a prismatic unbounded domain // “Lobachevskii Journal of Mathematics” Phys.-Math. Sci. 2021, V. 42 No. 15, p. 3606–3615.

10. S. Dzhamalov and M. Aliyev and Kh. Turakulov. "On a linear inverse problem for the three-dimensional Tricomi equation with nonlocal boundary conditions of periodic type in a prismatic unbounded domain. ", // "Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.", 2022, V. 42 No. 1, p. 1–12.

11. Lions J.L. and Magenes E., Inhomogeneous Boundary Value Problems and Their Applications, // M.: Mir, 1971.

12. Ladyzhenskaya O.A. Boundary value problems of mathematical physics. //Moscow, 1973, 407 p.

13. Vragov V.N. Boundary value problems for non-classical equations of mathematical physics. // Novosibirsk: NSU, 1983.