第 50 卷第 12 期
2023 年 12 月

湖南大学学报（自然科学版）
Journal of Hunan University（Natural Sciences）

Vol. 50. No. 12.
December 2023

Open Access Article

ON REfiNEMENTS OF THE ASYMPTOTIC EXPANSION OF THE CONTINUATION OF CRITICAL BRANCHING PROCESSES

Juraev Sh. Yu.
Kokand State Pedagogical Institute

Annotation. This article outlines ideas for improving the asymptotic expansion of the continuation of critical branching processes.

Keywords: critical branching, asymptotic expansion, Critical branching, factorial moments, asymptotic expansion.

For the probability of the continuation of critical branching processes, an asymptotic expansion is obtained under the assumption of the existence of factorial moments α_k for $k = 4,5,...,m,\ \ m < \infty$.

Let $Z_n,\ n = 0,1,2,...,$ be a branching process with discrete time and one type of particles and $Q_n = 1 - P_0(n)$ be the probability of the continuation of the process.

The asymptotic behavior of probability Q_n for discrete time was studied by A. N. Kolmogorov [1]. The results of A. N. Kolmogorov for processes with continuous time were obtained by B. A. Sevastyanov [2].

A literature review on the issues of limit theorems and local limit theorems, and in particular, refinement of the asymptotic expansion for probability Q_n, is briefly presented in the publication by S. V. Nagaev and R. Mukhamedkhanova [3].

In this abstract, we consider some refinements of the theorems proven in [3] on the asymptotic expansion for probability Q_n.

The following theorems were proven in [3] for critical branching processes $(A = 1)$, (see [3], pp. 96-97):

Theorem 1. If $A = 1, B > 0, C < \infty,$ then as $n \to \infty$:

$$Q_n = \frac{2}{Bn} + \left(\frac{4C}{3B^3} - \frac{2}{B}\right)\frac{\ln n}{n^2} + o\left(\frac{\ln n}{n^2}\right) \quad (1)$$

Theorem 2. If $A = 1, B > 0, D < \infty,$ then as $n \to \infty$:

Received: October 04, 2023 / Revised: October 30, 2023 / Accepted: November 18, 2023 / Published: December 28, 2023
About the authors：Juraev Sh. Yu.
Corresponding author- Email:

$$Q_n = \frac{2}{Bn} + \left(\frac{4C}{3B^3} - \frac{2}{B}\right)\frac{\ln n}{n^2} + \frac{4K}{B^2 n^2} + O\left(\frac{\ln n}{n^3}\right), \qquad (2)$$

where K is some constant dependent on the form of $F(x)$.

In [3], the authors reported that their methods for proving relations (1) – (2) are suitable for the case when factorial moments of a higher order exist.

Although not significant, The authors also considered the case when there are factorial moments $\alpha_k = F^{(k)}(1) < \infty$ for $k \geq 4$.

Now we proceed to the consideration of the case when there are factorial moments $\alpha_k < \infty$, where $k = 4,5,...,m,\ m < \infty$.

Theorem 3. If $A=1,\ B>0,\ \alpha_k < \infty,\ k \geq 4$, then as $n \to \infty$ for Q_n the following asymptotic formula holds :

$$Q_n = \left\{ \sum_{i=0}^{l} (-1)^i \left(\frac{2}{B}\right)^{2i+1} \frac{L_1^i \ln^i n}{n^{i+1}} + \left(\frac{2}{B}\right)^{2l+1} \frac{L_1^{l+1} \ln^{l+1} n}{n^{l+2}} + \right.$$
$$\left. + \left[\sum_{i=0}^{l} (-1)^{i+1} \left(\frac{2}{B}\right)^{2(i+1)} \frac{L_1^i \ln^i n}{n^{i+2}} \right] \left(K_m + M_{nm}\right) \right\} \left(1 + o\left(\frac{\ln n}{n}\right)\right) \qquad (3)$$

where $m \geq 4$,

$$K_m = 1 + L_1\left[1 + \frac{2c_1}{B} + O\left(\frac{1}{n}\right)\right] + \sum_{j=2}^{m-2} L_j\left(1 + I_j\right) - \sum_{j=2}^{m-2} L_j \zeta(j),$$

$$M_{nm} = \sum_{j=2}^{m-2} L_j R_{nj},\ R_{nj} = \frac{2}{B}\sum_{t=n}^{\infty}\frac{1}{t^j}\left(1 + o(1)\right),\ L_1 = \frac{B^2}{4} - \frac{C}{6}$$

$$I_j = \sum_{t=1}^{\infty} Q_t^j,\ \zeta(j) = \sum_{t=1}^{\infty}\frac{1}{t^j} - \textit{Euler zeta function},$$

and coefficients $L_i,\ i = 1,2,...,m-1$, depend only on factorial moments $\alpha_k < \infty,\ k = 2,3,...,m$, $c_1 = 0{,}577216... -$ is the Euler's constant, and parameter $l = 1,2,...,$ is defined below.

Comment

Parameter l in formula (3) determines the number of steps in the process of division with a remainder. From a practical point of view, it is more advantageous to assume that $l=1$ or 2, or 3, etc.

Obviously, parameter l allows us to determine the number of asymptotic terms in the asymptotic expansion for Q_n the probability of continuation of critical branching processes with discrete time.

It is easy to see that the expansion (3) contains $(l+1)(m-1)+1,\ (m\geq 4)$ of asymptotic terms, each of which has an explicitly defined constant coefficient, depending only on factorial moments $\alpha_k<\infty$, and the independent parameters l and m take the values of $l=1,2,...,\ m=4,5,....$

Proof of Theorem 3

If $\alpha_k<\infty,\ k=4,5,...,m,\ m<\infty,$ by Taylor's formula we have

$$Q_n=Q_{n-1}\left(1-\sum_{i=2}^{m}(-1)^{i-1}\frac{\alpha_i}{i!}Q_{n-1}^{i-1}+o\left(Q_{n-1}^{m-1}\right)\right) \tag{4}$$

Putting $b_n=\dfrac{1}{Q_n}$ from (4) we easily obtain

$$b_n=\frac{b_{n-1}}{1-\sum\limits_{i=2}^{m}(-1)^{i-1}Q_{n-1}^{i-1}\frac{\alpha_i}{i!}}=b_{n-1}+L_0+\sum_{j=1}^{m-1}L_jQ_{n-1}^{j}+o\left(Q_{n-1}^{m-1}\right) \tag{5}$$

where coefficients L_j depend on factorial moments $\alpha_k<\infty,\ k=2,3,...,m.$

These coefficients are found from the equality (5). For example, the coefficients L_0,L_1,L_2,L_3 have the form:

$$L_0=\frac{B}{2},\ \ L_1=\frac{B^2}{4}-\frac{C}{6},$$

$$L_2=\left[\frac{D}{24}-\frac{BC}{12}+\frac{B}{2}\left(\frac{B^2}{4}-\frac{C}{6}\right)\right],$$

$$L_3=\left[\frac{BD}{48}-\frac{E}{120}-\frac{C}{6}\left(\frac{B^2}{4}-\frac{C}{6}\right)\right]+\frac{B}{2}\left[\frac{D}{24}-\frac{BC}{12}+\frac{B}{2}\left(\frac{B^2}{4}-\frac{C}{6}\right)\right]$$

From (5) we obtain

$$b_n = 1 + L_0 n + L_1 \sum_{k=0}^{n-1} Q_k + \sum_{i=2}^{m-1} L_i \sum_{k=0}^{n-1} Q_k^i + o\left(\sum_{k=0}^{n-1} Q_k^{m-1} \right) \qquad (6)$$

Since at $n \to \infty$

$$Q_n = \frac{2}{Bn}\left(1 + o(1)\right), \qquad (7)$$

then

$$L_1 \sum_{k=0}^{n-1} Q_k = L_1 + L_1 \sum_{k=1}^{n-1} Q_k = L_1 \left[1 + \frac{2}{B} \sum_{k=1}^{n-1} \frac{1}{k}\left(1 + o(1)\right) \right] = L_1 \left[1 + \frac{2}{B}\left(c_1 + \ln n + O\left(\frac{1}{n}\right) + o(\ln n) \right) \right]$$

where $c_1 -$ is Euler's constant.

Putting $K = L_1\left(1 + \frac{2c_1}{B} + O\left(\frac{1}{n}\right)\right)$ we have

$$L_1 \sum_{k=0}^{n-1} Q_k = \frac{2L_1}{B} \ln n + o(\ln n) + K \qquad (8)$$

Further, we should investigate the sums in the right-hand side of the equality (6)

$$\begin{aligned} \sum_{j=2}^{m-2} L_j \sum_{k=0}^{n-1} Q_k^j &= \sum_{j=2}^{m-2} L_j + \sum_{j=2}^{m-2} L_j \sum_{k=1}^{n-1} Q_k^j = \sum_{j=2}^{m-2} L_j + \sum_{j=2}^{m-2} L_j I_j - \sum_{j=2}^{m-2} L_j \sum_{k=n}^{\infty} Q_k^j = \\ &= \sum_{j=2}^{m-2} L_j \left(1 + I_j\right) - \sum_{j=2}^{m-2} L_j \sum_{k=n}^{\infty} Q_k^j \end{aligned} \qquad (9)$$

where the series $I_j = \sum_{k=1}^{\infty} Q_k^j$ converges at each $j = 2,3,...,m-2,\ m = 4,5,....$

It's easy to see that with $n \to \infty$

$$\sum_{k=n}^{\infty} Q_k^j = \frac{2}{B} \sum_{k=n}^{\infty} \frac{1}{k^j} + o\left(\sum_{k=n}^{\infty} \frac{1}{k^j} \right) \qquad (10)$$

It is well known that the Riemann zeta function

$$\zeta(j) = \sum_{k=1}^{\infty} \frac{1}{k^j}, \quad \sum_{k=n}^{\infty} \frac{1}{k^j} = O\left(\frac{1}{n^{j-1}} \right) \qquad (11)$$

at each $j = 2,3,...$ has a finite value. From the table of integrals (see, for example, G. B. Dwight, "Tables of Integrals and Other Mathematical Formulas", "Nauka", M;,1973, p.16) we can obtain specific values of $\zeta(j)$.

Thus, from the equalities (19)-(11) it easily follows that

$$\sum_{j=2}^{m-2} L_j \sum_{k=0}^{n-1} Q_k^j = N_m - \sum_{j=2}^{m-2} L_j \left[\zeta(j) - R_{nj}\right]$$

where

$$N_m = \sum_{j=2}^{m-2} L_j \left(1 + I_j\right),\ R_{nj} = \frac{2}{B}\sum_{k=n}^{\infty}\frac{1}{k^j}\left(1 + o(1)\right) \qquad (12)$$

From relations (6), (9) - (12) it follows easily that

$$b_n = L_0 n + \frac{2L_1}{B}\ln n + o(\ln n) + K_m + M_{nm} \qquad (13)$$

where

$$K_m = 1 + K + N_m - \sum_{j=2}^{m-2} L_j \zeta(j),\ M_{nm} = \sum_{j=2}^{m-2} L_j R_{nj}.$$

From here we will easily have:

$$b_n = \left(L_0 n + \frac{2L_1}{B}\ln n + K_m + M_{nm}\right)\left(1 + o\left(\frac{\ln n}{n}\right)\right) \qquad (14)$$

Thus, we need to perform division with remainder in the following ratio

$$Q_n = \frac{1}{\frac{Bn}{2} + \frac{2L_1}{B}\ln n + K_m + M_{nm}} \cdot \left(1 + o\left(\frac{\ln n}{n}\right)\right) \qquad (15)$$

Dividing with remainder $l = 5$ steps we obtain

$$Q_n = \left\{\sum_{i=0}^{5}(-1)^i\left(\frac{2}{B}\right)^{2i+1}\frac{L_1^i \ln^i n}{n^{i+1}} + \frac{1}{b_n}\left\{\left(\frac{2}{B}\right)^{2\cdot 6}\frac{L_1^6 \ln^6 n}{n^6} + \left[\sum_{i=0}^{5}(-1)^{i+1}\left(\frac{2}{B}\right)^{2i+1}\frac{L_1^i \ln^i n}{n^{i+1}}\right]\left(K_m + M_{nm}\right)\right\}\right\}\left(1 + o\left(\frac{\ln n}{n}\right)\right) \qquad (16)$$

Obviously, the process of division with remainder can be continued indefinitely, but this approach is not advantageous from a practical point of view. We find it convenient to put $l = 1$ or 2 or 3, etc.

In formula (3), the parameter l allows us to determine the number of asymptotic syllables in the asymptotic expansion for Q_n.

The formula (3) can be easily proved by the method of induction.

Let in formula (3) $l = 1, m = 4$

$$Q_n=\left\{\sum_{i=0}^{1}(-1)^i\left(\frac{2}{B}\right)^{2i+1}\frac{L_1^i\ln^i n}{n^{i+1}}+\frac{1}{b_n}\left\{\left(\frac{2}{B}\right)^3\frac{L_1^2\ln^2 n}{n^3}+\left[\sum_{i=0}^{1}(-1)^{i+1}\left(\frac{2}{B}\right)^{2i+1}\frac{L_1^i\ln^i n}{n^{i+2}}\right](K_4+M_{4n})\right\}\right\}\left(1+o\left(\frac{\ln n}{n}\right)\right)$$

$$M_{n4}=L_2R_{n2}=\frac{2L_2}{B}\sum_{k=n}^{\infty}\frac{1}{k^2}\left(1+o(1)\right)=O\left(\frac{1}{n}\right)$$

$$Q_n=\left\{\frac{2}{Bn}-\left(\frac{2}{B}\right)^3\frac{L_1\ln n}{n^2}+\left(\frac{2}{B}\right)^3\frac{L_1^2\ln^2 n}{n^3}-\left(\frac{2}{B}\right)^2\frac{K_4}{n^2}+\left(\frac{2}{B}\right)^4\frac{K_4L_1\ln n}{n^3}\right\}\left(1+o\left(\frac{\ln n}{n}\right)\right)+$$

$$+\left\{O\left(\frac{1}{n^3}\right)+O\left(\frac{\ln n}{n^4}\right)\right\}\left(1+o\left(\frac{\ln n}{n}\right)\right)=\frac{2}{Bn}-\left(\frac{2}{B}\right)^3\frac{L_1\ln n}{n^2}-\left(\frac{2}{B}\right)^2\frac{K_4}{n^2}+\left(\frac{2}{B}\right)^3\frac{L_1^2\ln^2 n}{n^3}+$$

$$+\left(\frac{2}{B}\right)^4\frac{K_4L_1\ln n}{n^3}+O\left(\frac{1}{n^3}\right)$$

(17)

It follows from (17) that the result obtained at $l=1, m=4$ coincides with the result of Theorem 2.

Now, suppose that formula (3) holds when $l=k$. Putting $l=k$ in formula (3), rewriting it in a form similar to (16), we take one step of division with remainder. Then formula (3) is easily obtained at $l=k+1$ and we can say that this formula holds for any finite value of the parameter l.

Theorem 3 is proved.

List of used literature:

1. Adaptation Camacho M., Esteve V., Gisbert M. (2011). Delve into the deep: Learning potential in metaverses and 3D worlds (eLearning Papers).
2. Antonacci D.M., Modaress N. (2008). Envisioning the educational possibilities of user-created virtual worlds. AACE Journal, 16(2), 115-126.
3. Tu Chih-Hsiung, Blocher Michael and Roberts Gayle (2008) “Constructs for Web 2.0 learning environments: a theatrical metaphor” Educational Media International, Vol45, №4, pp. 253-269.
4. «Mathematical Literacy for Humanists» Herbert Gintis. USA, p.p.23-30.
5. Geometry Holme, A. Springer, Germany pp / 317-324